Iron, cobalt, nickel, copper, magnesium, and potassium have, up to the present, given negative results, but will be examined again.

XXI. "Formulæ for sn $8u$, cn $8u$, dn $8u$, in terms of sn u." By Ernest H. Glaisher, B.A., Trinity College, Cambridge. Communicated by J. W. L. Glaisher, M.A., F.R.S. Received June 16, 1881.

(Abstract.)

In Grunert's "Archiv der Mathematik und Physik," vol. xxxvi (1861), pp. 125–176, Baehr has given the formulæ for sn nu, cn nu, dn nu in terms of sn u for the cases $n=2, 3, 4, 5, 6, 7$. These expressions are reproduced in a tabular form in Cayley's "Treatise on Elliptic Functions," Art. 109.

The present paper contains the corresponding formulæ for the case of $n=8$. Denoting the numerators and common denominator of sn $4u$, cn $4u$, dn $4u$ by P, Q, R, S respectively, then the numerators and common denominator of sn $8u$, cn $8u$, dn $8u$ are respectively $2PQRS$, $S^4-2P^2S^2+k^2P^4$, $S^4-2k^2P^2S^2+k^2P^4$, $S^4-k^2P^4$; and the paper contains the values of these quantities and also of P^2, S^2, P^4, S^4 in terms of sn u, arranged in a tabular form.

XXII. "On Riccati's Equation and its Transformations, and on some Definite Integrals which satisfy them." By J. W. L. Glaisher, M.A., F.R.S., Fellow of Trinity College, Cambridge. Received June 16, 1881.

(Abstract.)

The memoir relates chiefly to the different forms of the particular integrals of the differential equation

$$\frac{d^2u}{dx^2}-a^2u=\frac{p(p+1)}{x^2}u \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

and to the evaluation of certain definite integrals which are connected with this equation.

Transforming (1) by assuming $u=x^{-p}v$ and putting $2p=n-1$, it becomes

$$\frac{d^2v}{dx^2}-\frac{n-1}{x}\frac{dv}{dx}-a^2v=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

and this may be transformed into Riccati's equation,

$$\frac{d^2v}{dz^2}-a^2z^{2q-2}v=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

by the substitution $x=\frac{1}{q}z^q$ where $q=\frac{1}{n}$.

It is well known that these equations admit of integration in a finite form if $p=$ an integer, $n=$ an uneven integer, and $q=$ the reciprocal of an uneven integer.

The memoir consists of an introduction and eight sections, the headings of which are: (1) Direct integration of the differential equation in series and connexion between the particular integrals; (2) New integration of the differential equation when $p=$ an integer; (3) Transformations of the original differential equation; Riccati's equation; (4) Special forms of the particular integrals in the cases in which the differential equations admit of integration in a finite form; (5) Evaluation of definite integrals satisfying the differential equations; (6) Symbolic forms of the particular integrals in the cases in which the differential equations admit of integration in a finite form; (7) Connexion with Bessel's Functions; (8) Writings connected with the contents of the memoir.

In the first section six particular integrals of the equation (1) are obtained and the relations between them are examined. When p is not an integer all the six particular integrals extend to infinity and the relations between them present no special peculiarity. When p is an integer two of the series terminate, and we thus obtain two particular integrals of (1), which contain a finite number of terms. The series terminate in consequence of the occurrence of zero factors in the coefficients of the terms, but if they be continued, zero factors occur also in the denominators, so that, after a finite number of terms, the series may be regarded as recommencing and extending to infinity. If the terminating series are supposed to recommence in this manner, so that all the series extend to infinity, then the relations between the particular integrals are the same as when p is not an integer; but if the series are supposed to terminate absolutely when the zero terms occur, the relations are quite different. As the finite portions of the particular integrals satisfy the differential equation, it is more natural to regard the series as terminating absolutely, and on this supposition the relations between the particular integrals exhibit a remarkable diversity of form, according as p is or is not an integer.

The second section contains what is believed to be a new form of the solution of (1) in the case of $p=$ an integer. It is shown that this equation is satisfied by the coefficient of h^{p+1} in the expansion of $e^{a\sqrt{(x^2+xh)}}$ in ascending powers of h. The six particular integrals given in the first section and the relations connecting them are also obtained by different expansions of this expression.

The third section contains the six particular integrals of (2) and (3), corresponding to those of (1), from which they are deduced.

The fourth section relates to the particular cases in which the differential equations admit of integration in a finite form. If a differential equation is satisfied by an infinite series, and if for certain values of a quantity involved in it the series terminates, then in this case we may present the integral in a different form by commencing the finite series at the other end and writing the terms in the reverse order. These reverse forms in the case of (1), (2), (3) are given in this section.

The fifth section contains the evaluations of the definite integrals—

$$\int_0^\infty e^{-x^m-\frac{a^2}{x^m}}dx, \qquad \int_0^\infty \frac{\cos bx}{(a^2+x^2)^n}dx,$$

where m denotes any real quantity and n any positive quantity. These integrals have been evaluated when m is of the form $\frac{-4i}{2i\pm 1}$, and when n is a positive integer, but the general formulæ are, the author believes, new. The results exhibit changes of form similar to those referred to in the account of the contents of the first section. Certain formulæ of Cauchy's and Boole's are also considered in this section.

The sixth section, which is the longest in the memoir, relates to the different symbolic solutions of (1), (2), (3) in the cases in which they are integrable in finite terms. In this section these symbolic solutions are derived from the definite integrals considered in the fifth section; and the various symbolic theorems to which they lead, by comparing different forms of the results, are examined. A great many symbolic solutions of these differential equations have been given by Gaskin, R. L. Ellis, Boole, Lebesgue, Hargreave, Williamson, Donkin, and others, and these are briefly noticed and connected with one another.

The seventh section relates to the connexion between the results contained in this memoir and the formulæ of Bessel's Functions. The equation (1) may, as is well known, be transformed into Bessel's equation

$$\frac{d^2w}{dx^2}+\frac{1}{x}\frac{dw}{dx}+\left(1-\frac{\nu^2}{x^2}\right)w=0$$

by the substitutions $u=x^{\frac{1}{2}}w$, $p+\frac{1}{2}=\nu$, $a^2=-1$, so that the theorems relating to the solutions of (1) have analogues in the solutions of Bessel's equation.

The eighth section contains a list of writings which are closely connected with the subject of the memoir, with short accounts of their contents.